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Some remarks on geodesics in gauge groups and harmonic maps

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Abstract. *We consider the gauge group $\mathcal{G}$ of automorphisms of a $U(N)$-principal bundle. We study the Euler equations for geodesics in $\mathcal{G}$ (or in associated spaces of connections). In particular, we show the existence and uniqueness of solutions for the Cauchy problem, and we give some examples of closed geodesics, connecting certain «harmonic elements» of $\mathcal{G}$ to the identity. Finally, we study the case of Riemann surfaces.*

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SUMMARY

Given $P \to M$, a smooth principal $SU(N)$-bundle over a compact Riemannian manifold M, we consider the «gauge group» $\mathcal{G}$ of smooth automorphisms of P.

Every connection A on $P \to M$ induces a weak right invariant Riemannian metric on $\mathcal{G}$, via right translation of the inner product:

$$\langle u, v \rangle = \int_M (d_A u, d_A v) \tag{0.1}$$

on the Lie algebra g of $\mathcal{G}$.

In §2, 3 we write down the Euler equations of geodesic motion in $\mathcal{G}$; in §6 we prove local existence and uniqueness of solutions for the associated Cauchy problem, in the hypothesis of irreducibility of A; in §4 we study certain conserved quantities.

In §7 we define «harmonic elements» of $\mathcal{G}$, with respect to the fixed connection A: they generalise harmonic maps $M \to U(N)$.

We then give two families of examples of closed geodesics in $\mathcal{G}$, connecting certain classes of such harmonic elements to the identity.

The first one consists of one-parameter subgroups of $\mathcal{G}$, associated to «harmonic subbundles» of the vector bundle $V \to M$, canonically associated to $P \to M$.

The second family is produced via a loop of connections with constant central curvature, in the case when M is a compact Riemann surface (cf. §8). (In particular we prove that Uhlenbeck's «extended solution» (cf. [20]) is a geodesic in the space of maps $M^2 \to U(N)$). This loop of connections is related to the theory of complete integrability, and Lax pairs with a complex parameter: cf. [10], [20], [22], [23].

Finally, again in the case when M is a Riemann surface, we show how to extract paths of holomorphic differentials and line bundles from geodesic paths in $\mathcal{G}$.

We believe that, excepted §8, 9 most of the material is just an application of a standard scheme to the case of the gauge group acting on connections: we could have therefore considered as well, for example, the case of the group of diffeomorphisms of a manifold, with its induced action on the space of Riemannian metrics, as in the theory of elasticity. In particular, the reader may find overlaps with [17], [18].

We have tried to avoid the use of infinite dimensional differential geometry: everything here is smooth, unless otherwise stated. The reader may find details on the «Lie group» structure of $\mathcal{G}$ in [3], [4], [15], [16], [18] (for example).

1. INTRODUCTION AND DICTIONARY

Let M be a compact Riemannian manifold, and let $P \to M$ be an $SU(N)$-principal bundle ($U(N)$ in §7, 8, 9).

Let $\mathcal{G}$ be the «gauge group» of smooth automorphisms of $P \to M$; if $V \to M$ is the complex hermitian vector bundle, canonically associated to $P \to M$ via the standard representation of $SU(N)$ in $\mathbb{C}^N$, then we have:

(1.1) $$\mathcal{G} \cong \{\text{smooth special unitary sections of } \mathrm{End}(V) \to M\}$$

The «Lie algebra» of $\mathcal{G}$, in a heuristic sense, but which can be made rigorous (cf. also §6), is:

(1.2) $$g = \{\text{smooth sections of the bundle } ad(P) \to M\}$$

(where $ad(P) \to M$ is the Lie algebra bundle associated to $P \to M$ via the adjoint representation); and we may identify:

(1.3) $$g \cong \{\text{smooth } su(N) - \text{sections of } \mathrm{End}(V) \to M\}$$

Let $\mathcal{A} = \mathcal{A}(P)$ be the space of connections on the principal bundle $P \to M$; $\mathcal{A}(P)$ is an affine space, modelled on the vector space $g \otimes T^*(M)$ of 1-forms, with coefficients in $su(N)$.

Each connection $A \in \mathcal{A}(P)$ induces, for each integer $k \geqslant 0$, an exterior differential:

(1.4) $$d_A : g \otimes \Lambda^k T^*(M) \to g \otimes \Lambda^{k+1} T^*(M)$$

We remark that:

(1.5) $$d_A d_A(\omega) = [K(A), \omega]$$

where $K(A) \in g \otimes \Lambda^2 T^*(M)$ is the curvature of the connection A.

Using local coordinates, we may identify a connection A with a 1-form $\tilde{A}$; we then have:

(1.6) $$K(A) = d\tilde{A} + 1/2[\tilde{A}, \tilde{A}]$$

Using the metric on M, and on $V \to M$, and the Killing form on $su(N)$, we may define Riemannian products (,) on each $g \otimes \Lambda^k T^*(M)$; consequently, we may define operators:

$$d_A^* : g \otimes \Lambda^{k+1} T^*(M) \to g \otimes \Lambda^k T^*(M)$$

which are adjoint to the d_A's, with respect to (,).

We can now define a «rough Laplacian» Δ_A :

(1.7) $$\Delta_A = d_A^* d_A + d_A d_A^* : g \otimes \Lambda^k T^*(M) \to g \otimes \Lambda^k T^*(M)$$

for each $k \geqslant 0$ (but we will use it only for $k = 0$).

The action of $\mathscr{G}$ on $P \to M$ induces an action:

$$\mathscr{G} \times \mathscr{A} \to \mathscr{A} \qquad (g, A) \mapsto g_*(A)$$

defined by:

$$d_{g_*(A)} v = g^{-1} d_A (g v g^{-1}) g \tag{1.8}$$

for each $v \in g$, and $g \in \mathscr{G}$ (but (1.8) is valid for $v \in g \otimes \Lambda^k T^*(M)$).

In local coordinates, if A is represented by the 1-form $A^\sim$, then we have:

$$g_*(A)^\sim = g^{-1} \tilde{A} g + g^{-1} \mathrm{d}g.$$

For $A \in \mathscr{A}(P)$, we indicate by $\mathscr{G}$. A the gauge orbit of A.

2. EULER'S EQUATION FOR GEODESICS IN $\mathscr{G}$

We now fix a connection $A \in \mathscr{A}(P)$ on $P \to M$.

We consider the (weak) inner product on g:

$$\langle u, v \rangle = \int_M (d_A u, d_A v) = -\int_M Tr(\Delta_A u, v) \tag{2.0}$$

We equip $\mathscr{G}$ with the (weak) (pseudo) right-invariant Riemannian metric induced from (2.0), via right translation; and we want to study the equations for geodesics in $\mathscr{G}$, with respect to this metric.

PROPOSITION 2.1. *Let $g_t : [a, b] \to \mathscr{G}$ be a path in $\mathscr{G}$. Then the following statements are equivalent:*

(i) g_t *is a geodesic;*

(2.1) *(ii)* $\mathrm{d}/\mathrm{d}t\, \Delta_A F + [\Delta_A F, F] = 0 \qquad \forall t \in [a, b]$

(where $F = g'g^{-1}$).

Proof. Let us consider the energy functional:

$$L^\sim(g) = \frac{1}{2} \int_a^b \langle g'g^{-1}, g'g^{-1} \rangle \, \mathrm{d}t =$$

$$= \int_a^b \mathrm{d}t \int_M (d_A (g'g^{-1}), d_A (g'g^{-1}))$$

on the space of smooth maps $g : [a, b] \to \mathcal{G}$ (smooth in a elementary sense: locally, on $U \subseteq M$, the map $g : [a, b] \times U \to SU(N)$ must be smooth).

We consider a 1-parameter variation:

$$g = g(s, t) : (-\epsilon, \epsilon) \times [a, b] \to \mathcal{G}$$

$$g(0, t) = g(t)\ \forall t; \quad g(s, a) = g(a) g(s, b) = g(b)\ \forall s;$$

and we compute the first variation of $L^\sim$.

$$\mathrm{d}/\mathrm{d}s\, L^\sim(g)|_{s=0} = \int_a^b \mathrm{d}t \int_M (\mathrm{d}/\mathrm{d}s\, d_A(g'g^{-1}), d_A(g'g^{-1}))|_{s=0} =$$

$$= \int_a^b \mathrm{d}t \int_M (\mathrm{d}/\mathrm{d}s\, (g'g^{-1})|_{s=0}, \Delta_A(g'g^{-1}))$$

Let $\mathrm{d}/\mathrm{d}s\, (g)|_{s=0} g^{-1} = v$.

$$\mathrm{d}/\mathrm{d}s\, L^\sim(g)|_{s=0} = \int_a^b \mathrm{d}t \int_M (\mathrm{d}/\mathrm{d}t\, (v) + [v, g'g^{-1}], \Delta_A(g'g^{-1})]) =$$

$$= \int_a^b \mathrm{d}t \int_M (v, -\mathrm{d}/\mathrm{d}t\, \Delta_A(g'g^{-1}) + [g'g^{-1}, \Delta_A(g'g^{-1})])$$

Therefore $\mathrm{d}/\mathrm{d}s\, L^\sim(g)|_{s=0} = 0$ for each variation $g^{\bullet} g^{-1} = v$ if and only if the right hand side satisfies the equation (2.1). ∎

We call (2.1) *(the first) Euler equation.*

Remark. Given $F : [a, b] \to g$, satisfying equation (2.1), we can recover the original map $g : [a, b] \to \mathcal{G}$, by solving, for each $x \in M$, the linear ordinary differential equation: $g' = Fg$; so eq. (2.1) is essentially equivalent, modulo right multiplication of g_t by a constant element of $\mathcal{G}$, to the geodesic equation in $\mathcal{G}$.

Similarly, the Cauchy problem:

$$\begin{cases} g : [a, b] \to \mathcal{G} \text{ geodesic} \\ g(a) = g_a, g'(a) = g'_a \end{cases}$$

Is equivalent to the Cauchy problem in g:

$$\begin{cases} d/dt\, \Delta_A F + [\Delta_A F, F] = 0 \\ F(a) = F_a, \quad F : [a, b] \to g. \end{cases}$$

3. EULER'S EQUATION FOR GEODESIC PATHS IN GAUGE ORBIT OF CONNECTIONS

We give now an equivalent formulation of eq. (2.1).

Remember we have fixed a connection A on $P \to M$, in order to give a Riemannian structure to $\mathcal{G}$. Moreover, this choice allows us to associate to each path $g_t : [a, b] \to \mathcal{G}$ a path of connections $A_t = (g_t)_* A$, in the same gauge orbit of A.

(Conversely, given a path of connections A_t, lying in the same gauge orbit, we can find, not uniquely if the connection is reducible, a path $g_t : [a, b] \to \mathcal{G}$, such that $A_t = (g_t)_* A$. This is due to a «local slice theorem» (cf. [3], [4])).

We remark that, using standard Sobolev spaces, as in §6, each $\mathcal{G}$-orbit may be considered as a submanifold of the affine space A, with tangent space:

$$T_B(\mathcal{G} \cdot B) = Im\{d_B : g \to g \otimes T^*(M)\} \tag{3.1}$$

(cf. [3], [4]); in particular, each gauge orbit inherits a Riemannian structure, and the following definition has a quite transparent geometrical meaning.

DEFINITION 3.1. We say that a path A_t of connections, lying in the same gauge orbit, is a geodesic path (in a $\mathcal{G}$-orbit), if for each t the acceleration vector A_t'' is orthogonal to the $\mathcal{G}$-action. ∎

We remark that this is equivalent, using (3.1), to the equation:

$$d_{A_t}{}^*(A_t'') = 0 \qquad \forall t \tag{3.2}$$

because $\ker d_{A_t}{}^* = (Im\, d_{A_t})^{\perp}$.

PROPOSITION 3.2. *Let A_t be a path of connections on a $SU(N)$-bundle $P \to M$, lying in the gauge orbit of a fixed connection A.*

Then the following statements are equivalent:

(i) *A_t is a geodesic path in the gauge orbit of A;*

(ii) *$A_t = (g_t)_*(A)$ where $g_t : [a, b] \to \mathcal{G}$ is a geodesic path in $\mathcal{G}$;*

(3.3) (iii) $d/dt\, (d_{A_t}{}^* A_t') = 0 \quad \forall t$

Proof. (3.2) $\Longleftrightarrow$ (3.3) is a trivial consequence of the fact that, $\forall v \in g \otimes T^*(M)$, we have $[v, *v] = 0$ (where $*$ is the Hodge operator:

(3.4) $$* : g \otimes \Lambda^k T^*(M) \to g \otimes \Lambda^{\dim M - k} T^*(M) \quad).$$

(ii) $\Longleftrightarrow$ (iii)

$$A_t' = d_{A_t}(g^{-1}g') = g^{-1}(d_A(g'g^{-1}))g$$

(by 1.8).

$$d_{A_t}{}^*(A_t') = g^{-1}(d_A{}^*d_A(g'g^{-1}))g = g^{-1}(\Delta_A(g'g^{-1}))g$$

So, we have:

$$\mathrm{d}/\mathrm{d}t\, d_{A_t}{}^*(A_t') = 0 \Longleftrightarrow \mathrm{d}/\mathrm{d}t\,(g^{-1}\Delta_A(g'g^{-1})g) = 0 \Longleftrightarrow$$
$$\Longleftrightarrow g^{-1}(\mathrm{d}/\mathrm{d}t\, \Delta_A(g'g^{-1}) - [g'g^{-1}, \Delta_A(g'g^{-1})])g = 0.$$ ∎

We call equation (3.3) *(the second) Euler equation.*

4. CONSERVED QUANTITIES AND MOMENT MAP

Definition 3.1 is natural. We can easily show that a path A_t in a $\mathcal{G}$-orbit of connections is geodesic if and only if it extremizes the appropriate energy (length) functional:

$$L^{\sim}(A_t) = \frac{1}{2}\int_a^b (A_t', A_t')\,\mathrm{d}t = L^{\sim}(g_t)$$

(Where g_t is an associated path in $\mathcal{G}$, as in prop. (3.2)).

Moreover, as in finite dimensional geometry, the geodesic flow on the tangent space to a gauge orbit $T(\mathcal{G}\cdot A)$ is canonically dual, via the L^2 inner product $(\,,)$ on $g \otimes T^*(M)$, to the hamiltonian flow on the cotangent bundle $T^*(\mathcal{G}\cdot A)$, with hamiltonian function the riemannian squared norm.

The $\mathcal{G}$-action of the space of connections $\mathcal{A}$ induces a $\mathcal{G}$-action on the cotangent bundle $T^*(\mathcal{A})$, and hence on $T^*(\mathcal{G}\cdot A)$:

$$(g, (A, T)) \mapsto (g_*(A), g^{-1}Tg)$$

This action is, by general principles in Hamiltonian mechanics, symplectic and almost hamiltonian (cf. [9], or any other book on geometric Hamitonian mechanics).

Dualizing, we get a moment map:

(4.1) $$P : T^*(\mathcal{A}) \to g^* \cong g \qquad P(A, T) = -\, d_A{}^*(T)$$

(we leave the computation to patient readers; it's not difficult, but it need some manipulations of the definitions; cf. also a similar computation in [11]).

In particular, $d_{A_t} {*A'_t}$ is invariant under the geodesic flow in $\mathscr{A}$, or in any $\mathscr{G}$-orbit.

Therefore, eq. (3.3) express the *conservation of momentum*.

The conservation of energy is given by the invariance of the arclength (A'_t, A'_t) under time evolution. Indeed we have: $d/dt\,(A'_t, A'_t) = 2(A''_t, A'_t) = 0$, because A'_t is tangent to the $\mathscr{G}$-orbit (by assumption), and A''_t is orthogonal to it (by definition of geodesic).

We can recover part of this conservation laws in eq. (2.1); (only part of them, because eq. (2.1) has less dependant variables).

PROPOSITION 4.1. *Let $F : [a, b] \to g$ be a solution of (2.1). Then:*

(i) *the functions* $\quad I^k(x) = Tr(\Delta_A F)^k(x) \qquad x \in M$

are constants of the motion;

(ii) *the energy* $\quad E(F) = \dfrac{1}{2}\displaystyle\int_M (\Delta_A(F), F)$

is a constant of the motion;

(iii) *if g and A_t are the corresponding geodesic paths in $\mathscr{G}$, and in $\mathscr{G}\cdot A$ ($F = g'g^{-1}$ and $A_t = (g_t)_* A$), so that, by eq. (3.2),*

$$d_{A_t} {*A'_t} = C \qquad \forall t, \qquad \textit{then we have:}$$

$$I^k(x) = Tr(C)^k, \quad E(F) = \frac{1}{2}\int_M (\Delta_A F, F) = (A'_t, A'_t).$$

Proof. (i) and (ii) follow from (iii). Moreover, as in the proof of prop. 3.2, $C = d_{A_t} {*A'_t} = g^{-1}\Delta_A f g$, $A'_t = g^{-1}(d_A F)g$, and g is unitary. ∎

Remarks. (i) (4.1) (i) easily follows from direct computation as well, because (2.1) is in the form of a Lax pair, and this produces the «isospectral» evolution of $\Delta_A F$.

(ii) The group of real positive functions on M, acts via conformal transformations on the space of metrics on M, but it does not act on the space of connections. Therefore, it changes the geodesic equations, by changing the «Hamiltonian», but it does not produce any moment map. When $\dim M = 2$ the action is anyway trivial (cf. §8).

5. HISTORICAL REMARKS

(A) The study of geodesics in a Lie group, equipped with a right (or left) invariant metric, dates back to Euler's work on the motion of a rigid body with one fixed point in $\mathbb{R}^3$. Indeed, this physical problem is mathematically equivalent to studying geodesics in $SO(3)$, equipped with a left invariant metric. This metric, obtainable through integration of the mass distribution of the rigid body, is actually the unique physical datum of the problem.

For the sake of fun, we translate the classical notations in our setting.

Let g_t be a geodesic path in $\mathcal{G}$, and let $(g_t)_*(A)$ be the corresponding geodesic path of connections. Then:

$\omega_b = g'g^{-1}$ is the velocity vector with respect to the body;

$\omega_s = g^{-1}g'$ is the velocity vector with respect to the space;

$\Delta_A : g \to g$ is the inertia operator (!);

$M_b = \Delta_A(g'g^{-1}) = \Delta_A F$ is the kinetic moment (with respect to the body);

$M_s = d_{A_t}{}^*A'_t = C$ is the kinetic moment (w.r.t. the space).

Finally, Euler's equations for the motion of a rigid body:

$$\mathrm{d}/\mathrm{d}t\, M_b + [M_b, \omega_b] = 0 \qquad \mathrm{d}/\mathrm{d}t\, M_s = 0$$

are our Euler's equations (2.1) and (3.3):

$$\mathrm{d}/\mathrm{d}t\, \Delta_A F + [\Delta_A F, F] = 0 \qquad \mathrm{d}/\mathrm{d}t\, (d_{A_t}{}^* A''_t) = 0.$$

(B) A second classic example of the study of geodesics in a group is the case:

$$\mathcal{G} = \{\text{volume preserving diffeomorphisms of a manifold } M\}$$

(M compact or with boundary); the geodesic flow in $\mathcal{G}$ describes the motion of an ideal incompressible fluid, moving in M. The corresponding equation is still due to Euler; but we actually ignore if he was conscious of the connection with his equation for the rigid body.

For a detailed analytic study of this case, see [7].

6. EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR THE CAUCHY PROBLEM

We want to study now the Cauchy problem:

$$\text{(6.1)} \qquad \begin{cases} \mathrm{d}/\mathrm{d}t\, \Delta_A F + [\Delta_A F, F] = 0 \\ F(0) = F_0 \qquad F : [a, b] \to g \end{cases}$$

All along this §, let A be an irreducibile connection on the principal $SU(N)$-bundle $P \to M$. In particular, the irreducibility of A implies that there are not covariant constant elements of g (the identity transformation is always covariant constant, but does not belong to g); and that the Laplacian $\Delta_A : g \to g$ is invertible.

We introduce the Sobolev spaces (cf. §1):

$$g^s = \{L^{2,s} - \text{sections of the bundle ad } P \to M\} \tag{6.2}$$

They are the Hilbert space completion of g with respect to the scalar product:

$$(u, v)_s = \Sigma_{j=1,,s}((\Delta_A)^j u, v)$$

The following statements are well known (cf. [3], [4], [15], [16]).

LEMMA 6.1. *Let $s > 1/2\, m$ ($m = \dim M$). Then*

(i) *g^s is a Banach Lie algebra:*

$K > 0$, $\forall u, v \in g^s$, we have $[u, v] \in g^s$ and

$$|\,[u, v]\,|_s \leqslant K(s)|\, u\,|_s\, |\, v\,|_s \tag{6.3}$$

(ii) *g^s is the Hilbert Lie algebra of the Hilbert Lie group:*

$$\mathcal{G}^s = \{L^{2,s} - \textit{automorphisms of } P \to M\}$$ ∎

LEMMA 6.2. *Let A be an irreducible connection on the principal $SU(N)$-bundle $P \to M$. Then we have:*

(i) $\ker(d_A : g \to g \otimes T^*(M)) = 0$

(ii) $\Delta_A = d_A {*d_A} : g^s \to g^{s-2}$ *is an isomorphism, $\forall s$;*

and $\Delta_A^{-1} : g^s \to g^s$ is a compact (hence continuous) operator. ∎

THEOREM 6.3. *Let A be an irreducible connection on the principal $SU(N)$-bundle $P \to M$.*

Then for each $F_0 \in g^s$, $s > 1/2\, m + 2$($m = \dim M$), the Cauchy problem (6.1) has a unique solution $F_t : \mathbb{R} \to g^s$.

In particular, if F_0 is smooth, then $F(t)$ is smooth for each t.

Proof. We write the system (6.1) in a different form: set $v = \Delta_A F \in g^{s-2}$; (6.1) is then equivalent to:

$$\begin{cases} \mathrm{d}/\mathrm{d}t\, v + [v, \Delta_A^{-1} v] = 0 \\ v(0) = v_0 \qquad v : [0, r] \to g^{s-2} \end{cases} \tag{6.4}$$

We write the system (6.4) in a more abstract form:

(6.5) $$d/dt\, v + f(v) = 0, \quad v(0) = v_0$$

where $f : X \to X$ (X Banach space).

In our case we have: $X = g^{s-2}$ and $f : g^{s-2} \to g^{s-2}$ is the non linear map:

(6.6) $$f(v) = [\Delta_A^{-1} v, v]$$ ∎

The method for solving (6.5) is exactly the same as the one normally used for solving the Cauchy problem for ordinary differential equations, starting from the transformation of (6.6) in the integral equation:

(6.7) $$v(t) = v_0 + \int_0^t f(v(\tau))\, d\tau$$

The following Proposition is a summary (with no pretension of generality) of Theorems 6.1.1 and 6.1.3 in [19], and of Theorems 5.1.1 and 5.6.1 in [13].

PROPOSITION 6.4. *Suppose that:*

(i) *The function f in (6.5) is of class C^1, and locally Lipschitz.*

(ii) *There exists a continuous non decreasing function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for each $t \geqslant 0$, the maximal solutions of the Cauchy problem:*

$$r' = g(r), \qquad r(0) = r0$$

exists for all positive times; and we have:

$$|f(u)| \leqslant g(|u|) \text{ for each } u \in X.$$

Then there exists a solutions $u : [0, \infty) \to X$ to the Cauchy problem (6.5). This solution is unique. ∎

It is quite easy to check, that, for each $s - 2 \geqslant 1/2\, m$ the function f given by (6.6) satisfies the assumptions (i) and (ii) of Proposition 6.4. This is a simple consequence of its quadratic nature, and of lemmas 6.1 and 6.2. It is indeed possible to choose as function g in (ii): $g(r) = C(s)r^2$. Therefore, by proposition 6.4, there exists a solution of the system 6.4 $u : [0, \infty) \to g^{s-2}$, and hence a solution $F : [0, \infty) \to g^s$ of (6.1). The existence for $t \leqslant 0$ is then a consequence of a simple argument of time inversion (cf. for example Remark 5.6.1 in [13]). ∎

COROLLARY 6.5. *The gauge group $\mathcal{G}$ is complete, in the sense of differential geometry, with respect to the pseudometric (0.1).* ∎

Notes. For an eventual Hopf-Rinow type theorem, cf. [6]; but Dowling's results don't look to be appliable to our situation.

For a detailed analytic study of a related problem (the study of geodesics in the group of volume preserving diffeomorphisms of a manifold M). cf. the article by Ebin and Marsden [7].

Remarks. The proof of Theorem 6.3 is quite easy. We support the idea that Euler's equation (2.1) should be considered as an ordinary differential equation, even if in a infinite dimensional space, with the Laplacian playing the role of an algebraic term, representing the curvature of $\mathcal{G}$; as it should be, being an equation describing geodesics.

If the connection A on $P \to M$ does not satisfy the conditions in lemma 6.2, so that the associated Laplacian Δ_A is not invertible, then we do not expect existence and uniqueness theorems to hold for the Cauchy System (6.1). Indeed, let us suppose we are looking for a solution of (6.1), expressed as a formal power series in $t: F(t) = \Sigma_{j \geq 0} t^j F_j$ with $F_j \in g$. Then (6.1) becomes:

$$\Delta_A F_{j+1} = 1/j + 1\ \Sigma_{0 \leq h \leq j} [F_{j-h}, \Delta_A F_h] \qquad \forall j \geq 0 \tag{6.8}$$

To solve (6.8) we have to proceed by induction on j; but, if the conditions of lemma 6.2 are not satisfied, then the Laplacian Δ_A is neither injective or surjective; therefore, at each stage, a choice of an F_j is not always possible, or, when possible, it is not unique.

In this case, trying to solve the Cauchy problem for the second Euler equation looks more promising:

$$\begin{cases} \mathrm{d}/\mathrm{d}t\,(d_{A_t} {*A'_t}) = 0 \quad A(0) = A_0,\ A'(0) = A'_0 \\ A'(t) \in Im\{d_{A_t} : g \to g \otimes T^*(M)\} \end{cases} \tag{6.9}$$

We indeed expect existence and uniqueness theorems to hold for (6.9); but rather than consider (6.9) directly (one could use the conservation of the momentum, to restrict the problem to integration of a vector field, but the main difficulty would be then the non-linear nature of the space on which it would be defined – a finite codimensional subspace of a gauge orbit), it may be more fruitful to restrict oneself to the study of based gauge group and gauge Lie algebras in a first place.

We are not experts in the subject, so we prefer to leave the problem open.

7. STATIONARY SOLUTIONS AND HARMONIC ELEMENTS

Let $P \to M$ be an $U(N)$-principal bundle over a compact Riemannian manifold M; Let $A \in \mathcal{A}(P)$ be a fixed connection on it; let $\mathcal{G}$ be the gauge group of smooth automorphisms of $P \to M$, and let g be its Lie algebra of smooth sections of $ad(P) \to M$.

DEFINITION. A solution $F : [a, b] \to g$ of the 1st Euler equation:

(2.1) $$\mathrm{d}/\mathrm{dt}\, \Delta_A F + [\Delta_A F, F] = 0$$

is called *stationary* if

(7.1) $$\mathrm{d}/\mathrm{d}t\, (\Delta_A F) = 0 = [F, \Delta_A F]$$

Examples. (i) Take F independant of time, satisfying the eq.:

$$\Delta_A (F) = \lambda(x)F(x).$$

(ii) In the case of geodesics in $SO(3)$, describing the motion of a rigid body with one fixed point in $\mathbb{R}^3$ (cf. §5), stationary solutions are just pure rotations of the body, with rotation axes the eigenvectors of the inertia operator. They are 1-parameter subgroups of $SO(3)$.

(iii) In the case of geodesics in the group of volume preserving diffeomorphisms of a manifold M, describing the motion of an incompressible ideal fluid in M, stationary solutions are just those which preserve the velocity vector of the fluid, in each point of M.

Let now B be another connection. Then $A - B$ is a $u(N)$-valued 1-form, and we may consider the L^2-energy:

(7.2) $$E : \mathscr{A}(P) \to \mathbb{R} \qquad E(B) = \frac{1}{2} \int_M (A - B, A - B)_{g \otimes T^*(M)}$$

DEFINITION. (i) We call B harmonic (with respect to A) if B is a critical point for E in its orbit.

(ii) We say that $g \in \mathscr{G}$ is *harmonic* (is an harmonic element, or an harmonic gauge), with respect to A, if g_*A is an harmonic connection (w.r.t. A).

PROPOSITION 7.1. *With the notations above:*

(i)	*B is harmonic (w.r.t. A)*	*if and only if*
(7.3)	$d_A{}^*(A - B) = 0$	*if and only if*
	A is harmonic (w.r.t. B).	
(ii)	*$g \in \mathscr{G}$ is harmonic (w.r.t. A)*	*if and only if*
	$d_A{}^*(A - g_*A) = 0$	*if and only if*

$$d_A{}^*(A - (g^{-1})_*(A)) = 0 \qquad \textit{if and only if}$$

(7.4) $$d_A{}^*(g^{-1} d_A g) = 0$$ ∎

COROLLARY 7.2. *$g \in \mathcal{G}$ is harmonic $\Leftrightarrow g^{-1}$ is harmonic.*

Proof. (i) B is harmonic with respect to A if and only if every variation B' of B, which is tangent to the gauge orbit, is orthogonal to the distance vector $A - B$. But the tangent space in B to the $\mathcal{G}$-orbit is given by (3.1) $Im\ d_B$; so B is harmonic w.r.t. A if and only if $A - B \in (Im\ d_B)^\perp = \text{Ker}\ d_B{}^*$.

But

(7.5) $$d_A{}^*(A - B) = d_A{}^*(A - B)$$

because, for each $X,\ Y \in g \otimes T^*(M)$, we have: $[X, *Y] + [Y, *X] = 0$; (where $*$ is the usual Hodge $*$ operator).

(ii) follows from (i), and from formula (1.8). ∎

Corollary 7.2 states that the involution:

$$J : \mathcal{G} \to \mathcal{G} \qquad J(g) = g^{-1}$$

restricts to the space of harmonic elements of $\mathcal{G}$. The fixed point set of J is:

(7.6) $$GR(\mathcal{G}) = \{g \in \mathcal{G} \mid g^2 = 1\}$$

(GR stays for Grassmannian); it may be identified with the space of subbundles of the complex vector bundle $V \to M$ canonically associated to $P \to M$ via the standard representation of $U(N)$ in $\mathbb{C}^N$. Indeed, $g^2 = 1$ if and only if we can write (uniquely)

(7.7) $$g = (p^\perp - p)$$

where $p = p^2$ is the hermitian projection operator onto a subbundle $\underline{p}$ of $V \to M$; and $p^\perp = (p^\perp)^2 = 1 - p$ is the hermitian projection operator onto the hermitian complement $\underline{p}^\perp$ of $\underline{p}$.

DEFINITION. We say that a subbundle $\underline{p}$ of $V \to M$ is harmonic with respect to a connection A if the associated element $g = p^\perp - p \in GR(\quad)$ is harmonic (w.r.t. A).

PROPOSITION 7.3. *Let $g \in \mathcal{G}, g^2 = 1$, so that $g = (p^\perp - p)$. Then we have:*

(1) $$E(g) = \frac{1}{2} \mid g_* A - A \mid^2 = 2 \int_M (d_A p, d_A p)$$

(2) *The following statements are equivalent.*
(i) g is harmonic.
(ii) g is a critical point of the energy (7.2), with respect to variations in $GR(\mathcal{G})$.
(7.8) (iii) $[\Delta_A p, p] = 0$

Proof. (1) follows from patient computations, which we prefer to omit.

(2) By (1), (ii) is equivalent to p being a critical point of

$$E(p) = 2\int (d_A p, d_A p)$$

in the space of p's. We allow variations of the form: $t \mapsto h^{-1}ph$, where $h : (-\epsilon, \epsilon) \to \mathcal{G}$, $h(0) = 1$. Therefore $p' = [p, v]$, with $v \in g$; and

$$E'(p) = 4\int (p', \Delta_A p) = -4\int (v, [p, \Delta_A p]$$

Therefore $E'(p) = 0 \ \forall v \in g \iff [p, \Delta_A p] = 0$.

By prop. 7.1, g is harmonic if and only if $d_A{}^*(g_*A - A) = 0$. We can use local coordinates on M, identifying the connection A, with a 1-form $\tilde{A} \in g \otimes T^*(M)$, with $d_A(\,.\,) = d(\,.\,) + [\tilde{A},\,.\,]$; then we have:

$$g_*(A) - A = g^{-1}\tilde{A}g + g^{-1}dg - \tilde{A};$$

write $g = 1 - 2p$:

$$g_*(A) - A = 2(2p\tilde{A}p - p\tilde{A} - \tilde{A}p + [p, dp]) =$$
$$= 2([p, dp + [\tilde{A}, p]]) = 2([p, d_A p]);$$

and we have:

$$d_A{}^*[p, d_A p] = [p, \Delta_A p].$$

Example. Let $P = M \times U(N)$, so that $\mathcal{G} = \{\text{smooth maps } M \to U(N)\}$, and $GR(\mathcal{G}) = \{\text{smooth maps } M \to G_k(\mathbb{C}^N), \text{ with } 1 \leqslant k \leqslant N-1\}$; and let A be the 0-connection, given by standard differentiation. Then an element in $\mathcal{G}$ (resp. in $GR(\mathcal{G})$) is harmonic if and only if it is *an harmonic map* $M \to U(N)$ (resp. $M \to G_k(\mathbb{C}^N)$). The equivalence of (i) and (ii) is nothing but the standard fact that an harmonic map into $G_k(\mathbb{C}^N)$ is harmonic if and only if it is harmonic as a map into $U(N)$. This is consequence of the fact that the inclusion map (7.7), viewed as a map $G_k(\mathbb{C}^N) \to U(N)$ is totally geodesic.

Let $g = (p^\perp - p) \in \mathcal{G}$ define a subbundle $\underline{p} \subseteq V \to M$. We associate to $\underline{p}$ the loop:

$$g_t : [0, 2\pi] \to \mathcal{G} \ t \mapsto (p^\perp + e^{it}p) = \exp(itp)$$

It is *a one parameter subgroup* of $\mathcal{G}$.

THEOREM 7.4. *Let $g \in \mathcal{G}$, $g^2 = 1$; and let $g_t = \exp(itp)$ be the associated 1-parameter subgroup of $\mathcal{G}$, as above.*

Then the following statements are equivalent.

(i) *$g = g_\pi$ is an harmonic element of $\mathcal{G}$.*

(ii) *$t \mapsto g_t$ is a (stationary) geodesic in $\mathcal{G}$, of length $\pi(2E(g))^{1/2}$.*

Proof. $g = g_\pi$ is harmonic $\Longleftrightarrow [\Delta_A p, p] = 0$

g_t is a geodesic $\Longleftrightarrow F = g'g^{-1}$ satisfies the first Euler equation (2.1):

$$d/dt\, \Delta_A F + [\Delta_A F, F] = 0$$

$$\Longleftrightarrow F = ip \text{ satisfies } [\Delta_A F, F] = 0.$$

Moreover the length of g_t is:

$$L(g_t) = 2\pi\, |\, d_A p \,| = 2\pi(1/2E(g))^{1/2} = \pi(2E(g))^{1/2}. \quad \blacksquare$$

DEFINITION. The *geodesic length spectrum* of $\mathcal{G}$ (with respect to the connection A) is the set of length of closed geodesics in $\mathcal{G}$.

The *energy spectrum* of harmonic elements of $\mathcal{G}$ (w.r.t. A is the set of energies of harmonic elements of $\mathcal{G}$. In particular, the energy spectrum of harmonic subbundles of $V \to M$ is the energy spectrum of harmonic idempotent elements g of $\mathcal{G}$.

COROLLARY 7.5. *The energy spectrum of harmonic subbundles of $V \to M$ is contained in $1/2\,\pi^2$ the square of the geodesic length spectrum of $\mathcal{G}$.*

Proof. By theorem 7.4, we can associate to each harmonic $g = (p^\perp - p)$ the closed geodesic $t \mapsto g_t = (p^\perp + e^{it}p)$. $\blacksquare$

Example. Let us consider, for simplicity, the case when M is a compact Riemann surface of genus h, $P = M \times U(N)$, and A is the zero connection. Then the construction in this § associates to each harmonic map $g = (p^\perp - p) : M \to G_k(\mathbb{C}^N)$ the geodesic in the group $\mathcal{G}$ of smooth maps $M \to U(N)$ $g_t = \exp(itp)$.

By a theorem of Killingback [12], $\pi_1(\mathcal{G}) = \pi_2(U(N)^{2h} \oplus \pi_3(U(N)) \cong \mathbb{Z}$.

Therefore we can associate a degree $\delta(g_t)$ to each map $g_t : S^1 \to \mathcal{G}$.

More over, we can define the topological degree $d(g)$ of a map $g : M^2 \to G_k(\mathbb{C}^N)$ as the algebraic degree of the induced map in cohomology: $f^* : H^2(G_k(\mathbb{C}^N), \mathbb{Z}) \cong \cong \mathbb{Z} \to \mathbb{Z} \cong H^2(M, \mathbb{Z})$.

PROPOSITION 7.6. *Let $g : M^2 \to G_k(\mathbb{C}^N)$, and let g_t be the associated 1-parameter subgroup of the space $\mathcal{G}$ of maps $M^2 \to U(N)$. Then the topological degree of g is equal to the topological degree of the loop g_t.*

Proof (sketch). To compute the degree of g_t, we view g_t not as a loop in $\mathcal{G}$, but as a map from M^2 into the «loop group» (cf. [2]) $\Omega U(N)$. The positive generator of the second cohomology group of $\Omega U(N)$ is the left invariant, symplectic 2-form S (cf. [2], [22]) induced via left translation by:

$$S(\xi, \chi) = \frac{1}{8i\pi^2} \int_{S^1} (\xi', \chi)$$

for ξ, χ in the Lie algebra of $\Omega U(N)$.

Therefore, doing some computation (cf. [21], [22]), we get:

$$\delta(g_t) = \int_M (g_t)_*(S) = \frac{1}{2\pi} \int_M |p^\perp \partial p|^2 - |p^\perp \partial p|^2 = d(g)$$

■

Remark. The constructions in this § may be generalized to other groups and symmetric spaces, rather than $U(N)$ and the complex Grassmannian.

As an example, let M be a compact Riemannian manifold of even dimension, and let A be a connection on its tangent space. Let $\mathcal{G}$ be the gauge group:

$$\mathcal{G} = \{\text{smooth orthogonal sections of } \mathrm{Aut}(M) \to M\}$$

We may define, via A, a right invariant metric on $\mathcal{G}$, as in § 1.

Let us consider the space:

$$\mathcal{T} = \{J \in \mathcal{G} \mid J^2 = -1\}$$

$\mathcal{T}$ may be described as the space of almost complex structures on M, which are compatible with its given Riemannian structure.

We consider the energy $E : \mathcal{G} \to \mathbb{R}$, and its restriction to $\mathcal{T}$; then we may define the spaces of harmonic elements of $\mathcal{G}$, and of harmonic almost complex structures, as above.

We associate to each almost complex structure J on M, the 1-parameter subgroup of $\mathcal{G}$:

$$t \mapsto g_t = \exp(tJ) = \cos t + J \operatorname{sen} t.$$

We have then a complete analogue of theorem 7.3.

PROPOSITION 7.7. *The following statements are equivalent.*

(i) $g_t = \exp(tJ)$ *is a (stationary) geodesic in* $\mathscr{G}$.

(ii) $J = g_{\pi/2}$ *is an harmonic almost complex structure.*

(7.10) (iii) $[\Delta_A J, J] = 0$

Note: eq. (7.10) was first shown to me by C. Wood. ∎

8. HARMONIC GAUGES ON RIEMANN SURFACES & LAX PAIRS

Let us take now $M = M^2$ compact Riemann surface, equipped with an hermitian metric, and $P \to M$ principal $U(N)$-bundle over M. Let $\mathscr{G} = \operatorname{Aut}(P)$ be the gauge group, and g its Lie algebra.

It is well-known that the unique topological invariant of a $U(N)$-principal bundle $P \to M^2$ is its first Chern class $c_1(P) \in H^2(M, \mathbb{Z}) \cong \mathbb{Z}$; or equivalently, the normalised 1st Chern class:

(8.1) $$\mu(P) = c_1(P)/\operatorname{rank} P$$

Let $\mathscr{A} = \mathscr{A}(P)$ be the space of connections on $P \to M$. We consider the subspace of $A(P)$:

(8.2) $$\mathscr{A}^\mu(P) = \{A \in \mathscr{A}(P) \mid * K(A) = -2i\pi\, \mu(P)\}$$

consisting of connections with constant central curvature $-2i\pi\mu(P)$.

The space $\mathscr{A}^\mu$ is not empty, because by a well known theorem of Narasimhan and Seshadri, it contains, in particular, stable bundles (cf. [3], [5]). Its «tangent space» at each point $B \in \mathscr{A}^\mu$ is given by:

(8.3) $$T_B(\mathscr{A}^\mu) = \{\ker d_B : g \otimes T^*(M) \to g \otimes \Lambda^2 T^*(M)\}$$

We remark that $\mathscr{A}^\mu$ is stable under the action of $\mathscr{G}$. Infinitesimally, this is expressed by comparing (8.3) and (3.1): indeed, for each $B \in \mathscr{A}^\mu$ we get, from (1.5) and the constant central curvature condition for the connection B, we get:

(8.4) $$d_B \circ d_B = 0$$

so that:

$$T_B(\mathscr{A}^\mu) = \ker d_B \supseteq \operatorname{Im} d_B = T_B(\mathscr{G} \cdot B)$$

For the sake of simplicity, if $B \in \mathscr{A}^\mu$, we say that B is a connection with μ-curvature.

As usual, we fix now a connection A on $P \to M$; moreover, we take A with μ-curvature. As in §7, we may form the L^2-energy: $E : A \to \mathbb{R}$

$$E(B) = \frac{1}{2} | A - B |^2 = \frac{1}{2} \int_M Tr(A - B) \wedge *(A - B) \tag{8.5}$$

Remarks. Because of the conformal invariance of the Hodge *-operator on 1-forms in dimension 2, E is invariant with respect with conformal changes of metric on M. Similarly, the usual right invariant metric on the gauge group $\mathcal{G}$, given by right translation of (0.1), is conformally invariant, and therefore the equations for geodesics in $\mathcal{G}$. What is not conformally invariant is the condition for a connection to have *constant* curvature, since it needs a volume form on M, in order to make sense. Anyway, once we have chosen the space $\mathcal{A}^\mu$, we will need only a conformal class of metrics on M^2, i.e. a complex structure, in the following.

As in §7, we define the spaces of harmonic connections, and of harmonic gauges (always with respect to the fixed μ-curvature connection A).

PROPOSITION 8.1. *Let B be a connection with μ-curvature on $P \to M^2$. Let us denote $\Phi = 1/2(A - B) \in g \otimes T^*(M)$, $Q = 1/2(A + B) \in \mathcal{A}(P)$.*

Then the following statements are equivalent:

(i) *B is harmonic (w.r.t. A).*

(ii) *for each $t \in [0, 2\pi]$ the connection:*

$$A_t = Q + \cos t\Phi + \text{sen}\, t * \Phi \qquad A_0 = A,\ A_\pi = B \tag{8.6}$$

has μ-curvature.

(iii) *for each $t \in [0, 2\pi]$ $A_{t+\pi}$ is harmonic with respect to A_t.*

(iv) *(Q, Φ) satisfies the following system of equations.*

$$\begin{cases} *(K(Q) + 1/2[\Phi, \Phi]) = -2i\pi\, \mu(P) \\ d_Q \Phi = d_Q(*\Phi) = 0 \end{cases} \tag{8.7}$$

Proof. We know, by prop. 7.1, that B is harmonic w.r.t. A if and only if $d_A *(A - B) = 0$. This is equivalent to $d_Q(*\Phi) = 0$. But we have:

$$K(A_t) = 1/2(1 + \cos t)K(A) + 1/2(1 - \cos t)K(B) + \\ + 1/2 \cos t\, d_B *(A - B)$$

Therefore $*K(A_t) = -2\pi i\mu \Longleftrightarrow d_B *(A-B) = 0,\ *K(A) = *K(B) = -2i\pi\mu$.
Moreover $*K(A) = *K(B) = -2i\pi\mu$ is equivalent to:

$$*(K(Q) + 1/2[\Phi, \Phi]) = -2i\pi\mu(P), \quad d_Q\Phi = 0$$

Moreover, the invariance of the system (8.7) under the S^1-action:

$$\Phi \mapsto \cos t\ \Phi + \operatorname{sen} t * \Phi$$

implies that we could have chosen as A, B any pair of connections $A_t, A_{t+\pi}$. ∎

Remarks. We have succeeded in representing the non-linear system (8.7) in the form of μ-curvature condition for a loop of connections A_t, or, in other words, in the form of a (local) compatibility condition for a loop of linear systems:

$$d_{A_t} v_t = 0 \qquad v : [0, 2\pi] \to g$$

This kind of representation is usually called Lax (or Zakharov-Shabat: cf. [23]) representation of the non linear system (8.7), and connects its study with the theory of completely integrable system, and soliton-type equations (cf. [23]). The representation is fundamental in analysing the solutions of (8.7).

For example, in the particular case, $P = M \times U(N)$, $\mu = 0$, $A = 0$, the loop of connections ((8.6) has been used to give complete geometrical descriptions of harmonic maps $S^2 \to U(N)$ (cf. [20], [22]) and $T^2 \to SU(2)$ (cf. [10]).

(2) The Hodge operator * verifies $(*)^2 = -1$ on 1-forms over M; it defines in this way a complex structure on the space of connections. The loop (8.6) is then a circle, of centre Q and ray Φ, lying in a complex affine subspace of $\mathscr{A}(P)$ of complex dimension 1. Conversely, given any such circle, lying in the space of μ-curvature connections, then its centre Q and any one of its rays Φ, are a solutions of (8.7).

The following definition has quite a transparent geometrical meaning, and it can be given the right variational justification, exactly as for def. 3.1.

DEFINITION. We say that a path A_t of μ-curvature connections on $P \to M$ is a geodesic path (in the space $\mathscr{A}^\mu$ of μ-curvature connections) if for each t the acceleration vector A_t'' is orthogonal to $\mathscr{A}^\mu$.

Because of (8.4), this is equivalent to saying that, for each t, A_t'' lies in: $\operatorname{Im} d_{A_t} : g \otimes \Lambda^2 T^*(M) \to g \otimes T^*(M)$.

PROPOSITION 8.2. *Let (Q, Φ) be solutions of (8.7) on $P \to M^2$. Let A_t be the associated loop of μ-curvature connections (8.6).*

Then the following statements are equivalent.

(i) *A_t is a geodesic loop in the space of μ-curvature connections.*

(ii) *A_t lies in the same $\mathcal{G}$-orbit for each t.*

(iii) *A_t is a geodesic loop in a $\mathcal{G}$-orbit of connections.*

(iv) $\exists g_t : [0, 2\pi] \to \mathcal{G}$ *geodesic loop such that*

(8.8) $(g_t)_*(A_0) = A_t$ *for each t.*

Proof. (iii) and (iv) are equivalent, as noticed in §3; moreover (iii) obviously implies (ii), which in turn is equivalent to:

for each t the velocity vector A'_t is tangent to the $\mathcal{G}$-action: i.e.

for each $t, A'_t \in \operatorname{Im} d_{A_t} : g \to g \otimes T^*(M)$.

But $A'_t = * A''_t$, so (ii) is equivalent to (i). Moreover (i) implies: (A_t remains in the same $\mathcal{G}$-orbit for each t) + ($A''_t \in \operatorname{Im} d_{A_t} \subseteq \operatorname{Ker} d_{A_t}$), which implies (iii). ∎

Remarks. (1) The «momentum» (cf. §4, 5) of the loop of connections A_t is:

$$d_{A_t} * (A'_t) = - 1/4[A - B, A - B] = - [\Phi, \Phi].$$

(2) In the case M is the Riemann sphere, the loop G_t in (8.8) always exists. This is due to the fact that on $\mathbb{C}P^1$, the space $\mathcal{A}^\mu$ of connections with μ-curvature always consists of a single $\mathcal{G}$-orbit (cf. [19]).

(3) Let us consider the case when $P = M^2 \times U(N)$, $\mu = 0$, $A = 0$; $\mathcal{G}$ is then isomorphic to the space of smooth maps from M^2 into $U(N)$. Then we can look at the loop (8.8) $g_t : S^1 \to \mathcal{G}$ as a map $G : M^2 \to \Omega U(N)$, where $\Omega U(N)$ is the «loop group» of $U(N)$ (cf. [2]). The map G has been called by Uhlenbeck [20] «extended solution»; it is a holomorphic map into the infinite dimensional Kahler manifold $\Omega U(N)$. Its degree is (cf. [22]) $1/8\pi$ the energy of the harmonic map $g_\pi : M^2 \to U(N)$

(4) Killingback has computed in [12] the homotopy groups of the gauge groups $\mathcal{G} = \{\text{smooth maps } M^2 \to U(N)\}$ (cf. also Prop. 7.6).

(5) The length of the loop A_t in (8.6) is (cf. theorem 7.4):

$$L(A_t) = \int_0^{2\pi} (A', A')^{1/2} = \pi | A - B | = \pi(2E_A(B))^{1/2}.$$

(6) A theorem ensuring the existence of harmonic connections in 0-curvature orbits, has been given by Gaveau (cf. [8]).

(7) We remark that the eventual existence of the loop G_t (8.8) implies the Morse instability of each connection $A_{t+\pi}$, harmonic with respect to A_t; (cf. [22] for a computation of the energy Hessian).

9. GEODESICS IN GAUGE GROUPS OVER RIEMANN SURFACES PRODUCE HOLOMORPHIC DATA

Let, like in the previous §, $P \to M^2$ be a principal $U(N)$-bundle over a compact Riemann surface $M = M^2$, A a fixed connection with μ-curvature on P, $\mathcal{G}$ the gauge group of automorphisms of P.

Let $g_t : [a, b] \to \mathcal{G}$ be a geodesic path in $\mathcal{G}$. We have shown the geodesic condition to be equivalent to the 2nd Euler equation:

$$(3.2) \qquad d_{A_t} *A_t'' = 0 \qquad \forall t$$

where $A_t = (g_t)*A$.

Equation (3.2) means that the 1-form $*A_t''$ is d_{A_t}-closed, for each t. But A_t has μ-curvature for each t, so that $d_{A_t} \circ d_A t = 0$, and we can perform an Hodge decomposition:

$$(9.1) \qquad *A_t'' = d_{A_t} v_t + H_t$$

$$(9.2) \qquad d_{A_t} H_t = 0 \; d_{A_t} *H_t = 0$$

where v_t and H_t are paths in g and $g \otimes T^*(M)$, respectively. Applying now to (9.2) the standard decomposition of $T^*(M) \otimes \mathbb{C}$ in $(1, 0)$ and $(0, 1)$ parts, relative to the complex structure on M, we get:

$$H_t = H_{\bar{z},t} + H_{z,t} \; d_{A_t} = \partial_{A_t} + \bar{\partial}_{A_t} \qquad \text{and:}$$

$$(9.3) \qquad \bar{\partial}_{A_t} H_t = 0$$

If $\mu = 0$ and the connection A is trivial ($A = j^{-1}\, dj$), then we may easily get from (9.3) a path of holomorphic 1-forms:

$$\Omega_{z,t} = (jg)^{-1} H_{z,t}(jg) \qquad \bar{\partial}\Omega_{z,t} = 0$$

If this is not the case, then the standard trick, in order to extract interesting objects from (9.3), is to apply $U(N)$-invariant polynomials to $H_{z,t}$ (cf. [11], or the Chern-Weil theory of characteristic classes).

For a proof of the following Proposition, cf. [11].

PROPOSITION 9.1. *Let g_t be a geodesic path in $\mathcal{G}$; and let $H_{z,t}$ be the path in $g \otimes T^*(M)$ obtained by the above procedure. Then:*

(i) *for each $k \geq 1$, the coefficient of λ^{N-k} in $\det(H_{z,t} - \lambda I)$ is a path of holomorphic k-differentials on M;*

(ii) *generically, the space*

$$M_t^{\sim} = \{(z, \lambda) \in T^*_{1,0}(M) \mid \det(H_{z,t} - \lambda I) = 0\}$$

is a N-fold, t-dependent branched covering of M; and the eigenspace bundle $L_t = \mathrm{Ker}(H_{z,t} - \lambda I) \subseteq V \to M$ is a path of holomorphic line bundles over $M_t^{\sim}$. ∎

Remarks. It is possible to reverse the whole procedure, finding back $H_{z,t}$ from $M_t^{\sim}$ and L_t (cf. [5], [11], and [22]).

This construction does not appear anyway to be useful in the description of geodesics in gauge groups over Riemann surfaces: its geometrical meaning is in fact obscure, and moreover the Cauchy problem (6.9) does not translate into a Cauchy problem for holomorphic differentials or holomorphic line bundles. We believe, anyway, that there are more things to say on the subject. As a very limited attempt to formulate questions, cf. next §.

10. SOME PROBLEMS AND IDEAS

(1) It is well known that the motion of a rigid body with one fixed point is a completely integrable Hamiltonian system. More generally, the geodesic flow on a finite dimensional semisimple Lie group, endowed with a right-invariant Riemannian metric, is completely integrable if and only if the associated Euler equation:

$$M' = [\omega, M]$$

(which is in the form of a Lax pair), may be written in the stronger form of a «Lax pair with spectral parameter»:

$$M'_\lambda = [\omega_\lambda, M_\lambda] \qquad \lambda \in \mathbb{C}$$

If so, all the eigenvalues of M_λ, for each λ, would provide conserved quantities for the motion, in sufficient number to ensure complete integrability (cf. [14]).

We ask the following problem: when is the geodesic flow in gauge groups a completely integrable infinite dimensional Hamiltonian system? The case of gauge groups of bundles over Riemann surfaces, as in §8, 9, is the one when the answer is more likely to be positive, because of the conformal invariance of the equations, and of an already large amount of recent results in related subjects. We have tried, but without any success, to construct new integrals of the motion, in addition to the ones described in §4, in the form of spectral functions of matrices or of differential operators.

(2) (Again in the case of Riemann surfaces).

Uhlenbeck has described (in [20]) a decomposition of an extended solution (8.8), which in §8 we proved to be a geodesic of $\mathcal{G}$, in terms of 1-parameter subgroups of $\mathcal{G}$, satisfying certain holomorphicity conditions.

It is possible to perform a similar procedure, using some sort of iteration of the construction in §7, for some class of closed geodesics in $\mathcal{G}$?

(3) Is it possible to describe in a simple way, for example in terms of the geodesic length spectrum and of the Laplace-spectrum of $P \to M$, the geodesic length spectrum of $\mathcal{G}$? (cf. Corollary 7.5 and [21]).

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